# ON THE TORSION OF A SOLID OF REVOLUTION UNDER AXISYMMETRICAL LOADING

## (O KRUCHENII TELA VRASHCHENIIA OSESIMMETRICHNOI NAGRUZKOI)

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B.L. ABRAMIAN
(Erevan)

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The paper gives the solution of some problems of torsion for shafts of circular cross-section of variable diameter. Cases are being studied in which the shaft has the form of a truncated circular cone twisted in the one case by a load distributed over a part of its lateral surface and in the other by concentrated moments also applied to the lateral surface of the shaft.

Another subject of study is the torsion problem for a hollow hemisphere by a load arbitrarily applied to its surface.

The solutions of these problems are represented by means of series in terms of Legendre's polynomials and of trigonometric functions.

The torsion problem of a conical shaft acted upon by moments applied to its end surfaces was first considered by Foeppl [1]; he obtained the solution of the problem with the aid of a potential function. The torsion of a conical shaft among other solids of revolution has been studied also by Lokshin [2]. The solution of the torsion problem for a hollow conical shaft was given by Panarin [3]. The torsion of a conical shaft with the twisting load applied to its lateral surface according to a power law was investigated by the author [4]. The torsion problem of a sphere under concentrated moments is considered in the contributions of Snell [10] and Huber [11]. Some further problems dealing with torsion of shafts of variable cross-section have been studied in spherical coordinates in a publication by Solianik-Krassa [5].

**1. Statement of the problem.** The torsion problem of a solid of revolution under axisymmetrical loading will be treated in the following by means of a displacement function $\psi(r, z)$, which satisfies the differential equation

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{3}{r}\frac{\partial\Psi}{\partial r}+\frac{\partial^2\Psi}{\partial z^2}=0 \tag{1.1}$$

in the region of the axial section of the solid and the normal derivative

$$\frac{\partial\Psi}{\partial\nu}=\frac{1}{Gr}S_\nu(r,z), \qquad (S_\nu(r,z)=\tau_{r\varphi}\cos(r,\nu)+\tau_{z\varphi}\cos(z,\nu)) \tag{1.2}$$

is given along the boundary line of the axial section. $S_\nu(r, z)$ represents the projection of the complete shear stress on the normal to the contour of the axial section, while

$$\begin{aligned}\tau_{r\varphi}&=Gr\frac{\partial\Psi}{\partial r}, &\qquad \cos(r,\nu)&=\frac{dr}{d\nu}=\frac{dz}{ds}\\ \tau_{\varphi z}&=Gr\frac{\partial\Psi}{\partial z}, &\qquad \cos(z,\nu)&=\frac{dz}{d\nu}=-\frac{dr}{ds}\end{aligned} \tag{1.3}$$

The twisting moment of the external forces, distributed over the free surface of the solid of revolution from one point of the axis of the solid to any cross-section of the latter, determined by the distance $s$ along the generatrix of the solid, equals

$$M(s)=2\pi\int_0^s r^2(s)\,S_\nu[r(s),\ z(s)]\,ds \tag{1.4}$$

Carrying out the integration along the entire length of the generator of the shaft from one point of the axis to the other, we arrive from (1.4) at a relation representing the equation of equilibrium of moments, which produce the twisting of the shaft.

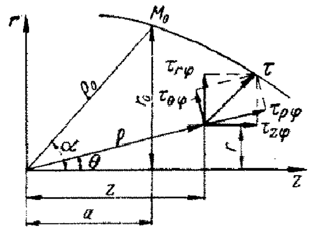


**Fig. 1.**

Transformation to spherical coordinates $\rho$, $\theta$, $\phi$, and to polar coordinates $\rho$, $\theta$, or $t$, $\xi$, in the $rz$-plane, by means of the formulas

$$\begin{aligned}r&=\rho\sin\theta=\rho_0e^t\sqrt{1-\xi^2},\ \xi=\cos\theta=\frac{z}{\sqrt{r^2+z^2}}\\ z&=\rho\cos\theta=\rho_0e^t\xi, \qquad t=\ln\frac{\rho}{\rho_0}=\ln\frac{\sqrt{r^2+z^2}}{\rho_0}\end{aligned} \tag{1.5}$$

where $\rho$ is the radius vector of a point in the axial section of the solid of revolution (Fig. 1), while $\theta$ is the angle between the radius vector $\rho$ and the $z$-axis and $\rho_0$ represents the distance of an arbitrarily chosen point $M_0$ on the contour of the axial section of the solid, replaces Equation (1.1) by

$$\frac{\partial^2\Psi^*}{\partial t^2}+(1-\xi^2)\frac{\partial^2\Psi^*}{\partial\xi^2}+3\frac{\partial\Psi^*}{\partial t}-4\xi\frac{\partial\Psi^*}{\partial\xi}=0 \tag{1.6}$$

with the notation

$$\Psi[r(t,\xi),\ z(t,\xi)] = \Psi^*(t,\xi) \tag{1.7}$$

In the new coordinates we have the expressions

$$\tau_{\rho\varphi} = \tau_{t\varphi} = G\sqrt{1-\xi^2}\,\frac{\partial\Psi^*}{\partial t}, \qquad \tau_{\theta\varphi} = \tau_{\xi\varphi} = -G(1-\xi^2)\,\frac{\partial\Psi^*}{\partial\xi} \tag{1.8}$$

for the stress components.

**2. General solution of Equation (1.6).** Substituting into (1.6) the formula

$$\Psi^*(t,\xi) = T(t)\,\varphi(\xi) \tag{2.1}$$

and separating the variables for the purpose of determining the functions $T(t)$ and $\phi(\xi)$, we obtain two groups of equations

$$(1-\xi^2)\varphi''(\xi) - 4\xi\varphi'(\xi) + \lambda^2\varphi(\xi) = 0, \qquad T''(t) + 3T'(t) - \lambda^2 T(t) = 0 \tag{2.2}$$

$$(1-\xi^2)\varphi''(\xi) - 4\xi\varphi'(\xi) - \lambda^2\varphi(\xi) = 0, \qquad T''(t) + 3T'(t) + \lambda^2 T(t) = 0 \tag{2.3}$$

The solution of Equations (2.2) is given by the functions

$$\varphi_n(\xi) = \frac{d}{d\xi}\,[AP_n(\xi) + BQ_n(\xi)] \tag{2.4}$$

$$T_n(t) = e^{-3/2t}\left[C\sinh\frac{(2n+1)t}{2} + D\cosh\frac{(2n+1)t}{2}\right] \tag{2.5}$$

where $P_n(\xi)$ is Legendre's spherical function of the first kind and $n$th order [6 ], while $Q_n(\xi)$ is a function of the second order and $\lambda$ is connected with $n$ by means of the equation

$$\lambda^2 + 2 - n^2 - n = 0 \tag{2.6}$$

We use only the solutions for which $n$ is a positive natural number; then the functions $P_n(\xi)$ and $Q_n(\xi)$ will be Legendre's polynomials of the first and second order, respectively. Since the function $Q'(\xi)$ approaches infinity when $\xi = 1$, the coefficient $B$ is to be taken equal to zero in the solution for solid shafts.

The solution of Equations (2.3) is given by the functions

$$\varphi_k(\xi) = \frac{d}{d\xi}\,[AP_{-1/2+\mu_k i}(\xi) + BQ_{-1/2+\mu_k i}(\xi)] \tag{2.7}$$

$$T_k(t) = e^{-3/2t}(C\sin\mu_k t + D\cos\mu_k t) \qquad \left(\mu_k = \sqrt{\lambda_k^2 - \tfrac{9}{4}}\right) \tag{2.8}$$

where $P_{-1/2+\mu_k i}(\xi)$ and $Q_{-1/2+\mu_k i}(\xi)$ are conical functions [6,7]. The coefficient $B$ in (2.7) is again to be taken equal to zero in the solution for solid shafts.

Finally, we note that the functions

$$e^{-3t},\ 1,\ \frac{\xi}{1-\xi^2}+\frac{1}{2}\ln\frac{1+\xi}{1-\xi} \tag{2.9}$$

are also particular solutions of Equation (1.6).

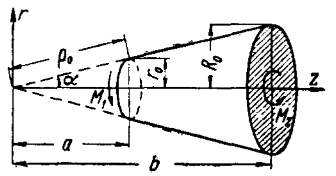


Fig. 2.

The general solution of Equation (1.6) is to be set up with the aid of the functions (2.4), (2.5), (2.7), (2.8) and (2.9).

**3. Torsion of a conical shaft under arbitrary loading of its lateral surface by shear stresses.** Consider a shaft of circular cross-section of variable diameter, having the shape of a truncated cone (Fig. 2). Assume that this shaft is being twisted by concentrated moments $M_1$ and $M_2$, applied to the plane end surfaces and an arbitrary loading applied to its lateral surface. Using the relation (1.4) we have

$$M_1 = -2\pi\int_0^{r_0} r^2\tau_{\varphi z}(r, a)\,dr, \qquad M_2 = 2\pi\int_0^{R_0} r^2\tau_{\varphi z}(r, b)\,dr \tag{3.1}$$

On the lateral surface of the shaft the stresses

$$\tau_{\xi\varphi}(t, \xi_1) = S_\nu[r(t, \xi_1),\ z(t, \xi_1)] = f_1(t) \tag{3.2}$$

are given, where $\xi_1 = \cos\alpha$, while $f_1(t)$ is a function sectionwise continuous in the interval $(0, t_1)$ of bounded variation in this interval.

In accordance with (2.7), (2.8) and (2.9) we assume the solution for $\Psi^*(t, \xi)$ to be of the form

$$\Psi^*(t, \xi) = A_0\rho_0^{-3}e^{-3t} + B_0P'_{-1/2}(\xi)e^{-1/2t} + \sum_{k=1}^{\infty} B_kP'_{-1/2+\mu_k i}(\xi)T_k(t) \tag{3.3}$$

$$(\xi_1 < \xi < 1,\ 0 < t < t_1)$$

where

$$T_k(t) = e^{-1/2t}\cos\mu_k t, \qquad \mu_k = \frac{k\pi}{t_1} \qquad (k = 1, 2, \ldots) \tag{3.4}$$

$$t_1 = \ln \frac{\sqrt{R_0^2 + b^2}}{\rho_0} = \ln \frac{b}{a}, \qquad \rho_0 = \sqrt{r_0^2 + a^2}$$

The functions $T_k(t)$ are orthogonal with the weight $e^{3t}$, and they satisfy the relations

$$\int_0^{t_1} e^{3/2 t} T_k(t)\,dt = 0, \qquad \int_0^{t_1} e^{3t} T_k(t)\, T_p(t)\,dt = \begin{cases} 0 & (p \neq k) \\ \frac{1}{2} t_1 & (p = k) \end{cases} \tag{3.5}$$

In cylindrical coordinates the displacement function $\Psi(r, z)$ assumes the form

$$\Psi(r, z) = \frac{A_0}{(r^2 + z^2)^{3/2}} + \frac{B_0 \rho_0^{3/2}}{(r^2 + z^2)^{3/4}} P'_{-1/2}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) + \sum_{k=1}^{\infty} B_k P'_{-1/2+\mu_k i}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) T_k\left(\ln \frac{\sqrt{r^2 + z^2}}{\rho_0}\right) \tag{3.6}$$

Computing the stresses $\tau_{\phi z}$ and $\tau_{\xi\phi}$ we find

$$\tau_{\varphi z}(r, z) = Gr \frac{\partial \Psi}{\partial z} = Gr \left\{ -3A_0 \frac{z}{(r^2 + z^2)^{5/2}} + B_0 \rho_0^{3/2} \left[ \frac{r^2}{(r^2 + z^2)^{9/4}} P''_{-1/2}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) - \frac{3z}{2(r^2 + z^2)^{7/4}} P'_{-1/2}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)\right] + \sum_{k=1}^{\infty} B_k \left[ \frac{r^2}{(r^2 + z^2)^{3/2}} P''_{-1/2+\mu_k i}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) \times T_k\left(\ln \frac{\sqrt{r^2 + z^2}}{\rho_0}\right) + \frac{z}{r^2 + z^2} P'_{-1/2+\mu_k i}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) T_k'\left(\ln \frac{\sqrt{r^2 + z^2}}{\rho_0}\right)\right]\right\} \tag{3.7}$$

$$\tau_{\xi\varphi}(t, \xi) = -G(1 - \xi^2) \frac{\partial \Psi^*}{\partial \xi} = -G(1 - \xi^2) \left\{ B_0 e^{-3/2 t} P''_{-1/2}(\xi) + \sum_{k=1}^{\infty} B_k P''_{-1/2+\mu_k i}(\xi)\, T_k(t) \right\} \tag{3.8}$$

Using the expansions

$$f_1(t) = g_0 e^{-3/2} + \sum_{k=1}^{\infty} g_k T_k(t) \qquad (0 < t < t_1) \tag{3.9}$$

where

$$g_0 = \frac{1}{t_1} \int_0^{t_1} e^{3/2 t} f_1(t)\,dt, \qquad g_k = \frac{2}{t_1} \int_0^{t_1} e^{3t} f_1(t)\, T_k(t)\,dt \tag{3.10}$$

we obtain from condition (3.2) the expressions

$$B_0 = -\frac{g_0}{G(1 - \xi_1^2) P''_{-1/2}(\xi_1)}, \qquad B_k = -\frac{g_k}{G(1 - \xi_1^2) P''_{-1/2+\mu_k i}(\xi_1)} \tag{3.11}$$

Satisfying the first of conditions (3.1) and eliminating from the expression thus obtained the coefficients $B_0$ and $B_k$, we obtain

$$A_0 = \frac{M_1}{2\pi G c_0} - \frac{\rho_0^3 \sin^2\alpha}{G c_0}\left[\frac{3}{2}\sum_{k=1}^{\infty}\frac{g_k}{\mu_k^2+\frac{9}{4}} + \frac{2}{3}g_0\right] \tag{3.12}$$

where use is made of the notation

$$c_0 = 2 - 3\cos\alpha + \cos^3\alpha \tag{3.13}$$

and the equality $\frac{\partial}{\partial r}\left[\frac{r^4}{(r^2+z^2)^{1/2}} T_k'(t)\, P''_{-1/2+\mu_k i}(\xi)\right]$

$$= -\left(\mu_k^2 + \frac{9}{4}\right)\frac{r^3}{r^2+z^2}\left[\frac{r^2}{(r^2+z^2)^{1/2}} P''_{-1/2+\mu_k i}(\xi)\, T_k(t) + z P'_{-1/2+\mu_k i}(\xi)\, T_k'(t)\right] \tag{3.14}$$

Satisfying the second of conditions (3.1) and eliminating from the resulting expression the coefficients $A_0$, $B_0$ and $B_k$, we obtain an equality connecting the Fourier coefficients $g_0$ and $g_k$ with the moments $M_1$ and $M_2$.

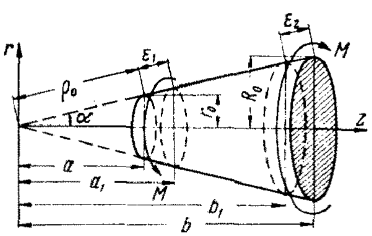

Fig. 3.

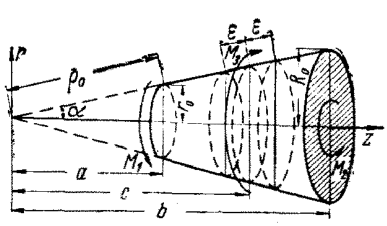

Fig. 4.

This equality represents the equilibrium equation for the moments which produce the twisting of the shaft; it has the following form:

$$\frac{M_1+M_2}{2\pi} + \rho_0^3\sin^2\alpha\left\{\frac{2}{3}g_0\left(e^{3/2 t_1} - 1\right) - \frac{3}{2}\sum_{k=1}^{\infty}\frac{g\ \left[1+(-1)^{k+1}e^{3/2 t_1}\right]}{9/4+\mu_k^2}\right\} = 0 \tag{3.15}$$

**4. Special cases.** *1. Truncated solid cone twisted by moments $M_1$ and $M_2$ applied at the ends $z = a$ and $z = b$.* Since in this case the lateral surface is free of loads, we must have $g_0 = g_k = 0$. The equilibrium equation (3.15) assumes the known form $M_1 = -M_2 = M$. From (3.11) and (3.12 we obtain the constants of integration

$$B_0 = B_k = 0, \qquad A_0 = \frac{M}{2\pi G(2 - 3\cos\alpha + \cos^3\alpha)} \tag{4.1}$$

This leads to the Foeppl solution [1].

*2. Truncated cone twisted by loading applied to two regions of the lateral surface of the shaft (Fig. 3).* The plane bases of the cone are assumed to be free of loading (the analogous problem for the cylinder was considered by Filon [8]). In this case

$$M_1 = M_2 = 0 \tag{4.2}$$

Suppose the loading is distributed over the lateral surface of the shaft in accordance with the law

$$\tau_{\xi\varphi}(t,\ \xi_1) = f_1(t) = \begin{cases} p_1 & (0 < t < \varepsilon_1), \quad (\varepsilon_1 = \ln a_1 / a) \\ -p_2 & (t_1 - \varepsilon_2 < t < t_1), \quad (\varepsilon_2 = \ln b / b_1) \end{cases} \tag{4.3}$$

Using Expression (3.10) we find

$$g_0 = \frac{2}{3t_1}\left[p_1\left(e^{3/2\varepsilon_1} - 1\right) - p_2\left(e^{3/2 t_1} - e^{3/2(t_1 - \varepsilon_2)}\right)\right] \tag{4.4}$$

$$g_k = \frac{2}{t_1\left(\frac{9}{4} + \mu_k^2\right)}\left\{p_1\left[e^{3/2\varepsilon_1}\left(\mu_k \sin\mu_k\varepsilon_1 + \frac{3}{2}\cos\mu_k\varepsilon_1\right) - \frac{3}{2}\right] - \right.$$
$$\left. - p_2\left[\frac{3}{2}(-1)^k e^{3/2 t_1} - e^{3/2(t_1-\varepsilon_2)}\left(\mu_k \sin\mu_k(t_1 - \varepsilon_2) + \frac{3}{2}\cos\mu_k(t_1 - \varepsilon_2)\right)\right]\right\} \tag{4.5}$$

The displacement function is determined in this case by Expression (3.3), where the coefficients $A_0$, $B_0$ and $B_k$ must be computed by means of Formulas (3.11) and (3.12), with relations (4.2), (4.4) and (4.5) taken into account.

**3. Truncated cone twisted by concentrated moments (Fig. 4).** For the case of an infinitely long cylinder such a problem has been studied by Timpe [9].

Let the shaft be twisted by concentrated moments $M_1$ and $M_2$ applied to the plane end surfaces of the shaft, and a moment $M_3$ applied to its lateral surface.

We represent the lateral torque by a loading distributed over a region of width $2\epsilon$ of the lateral surface of the shaft according to the law

$$\tau_{\xi\varphi}(t,\ \xi_1) = f_1(t) = \begin{cases} 0 & (0 < t < t_0 - \varepsilon,\ t_0 + \varepsilon < t < t_1), \quad (t_0 = \ln(c / a)) \\ -p & (t_0 - \varepsilon < t < t_0 + \varepsilon), \quad (t_1 = \ln(b / a)) \end{cases} \tag{4.6}$$

Then the torque $M_3$ is determined according to (1.4) by the formula

$$M_3 = -2\pi\rho_0^3 \sin^2\alpha \int_0^{t_1} e^{3t} f_1(t)\,dt = \frac{4}{3}\pi\rho_0^3 \sin^2\alpha e^{3t_0} p \sinh 3\varepsilon \tag{4.7}$$

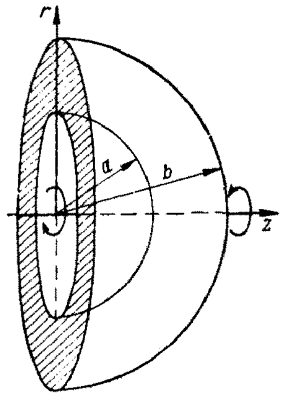


Fig. 5.

Having computed $g_0$ and $g_k$ in accordance with (4.6) and (3.10), using relation (4.7) and passing to the limit at $\epsilon \to 0$, we find for these coefficients the following expressions:

$$g_0 = -\frac{M_3 e^{-3/2 t_0}}{2\pi t_1 \rho_0^3 \sin^2\alpha} \tag{4.8}$$

$$g_k = -\frac{M_3 e^{-3/2 t_0} \cos\mu_k t_0}{t_1 \pi \rho_0^3 \sin^2\alpha} = 2g_0 \cos\mu_k t_0$$

The displacement function is determined in this problem by Expression (3.3), where the coefficients $A_0$, $B_0$ and $B_k$ have the values (3.11) and (3.12), taking into account the values determined by Formula (4.8).

**5. Torsion of a hemisphere.** Consider a solid of revolution having the shape of a hollow hemisphere (Fig. 5), and assume that it is being twisted by a loading arbitrarily applied to the free surface of the solid, so that

$$\tau_{\xi\varphi}(t, 0) = f_1(t) \qquad (0 < t < t_1) \quad (t_1 = \ln(b/a)) \tag{5.1}$$

$$\tau_{t\varphi}(0, \xi) = f_2(\xi), \qquad \tau_{t\varphi}(t_1, \xi) = f_3(\xi) \quad (0 < \xi < 1) \tag{5.2}$$

The functions $f_i$ are here sectionwise continuous with bounded variation in the respective intervals. The solution of Equation (1.6) is, in accordance with (2.4), (2.5), (2.7), (2.8) and (2.9), to be taken in this case in the form

$$\Psi^*(t, \xi) = A_0 e^{-3t} + e^{-3/2 t} \sum_{k=1}^{\infty} [A_k \sinh\alpha_k t + C_k \cosh\alpha_k t]\, \varphi_{2k+1}(\xi) + \tag{5.3}$$

$$+ B_0 e^{-3/2 t} P'_{-1/2}(\xi) + \sum_{k=1}^{\infty} B_k P'_{-1/2+\mu_k i}(\xi)\, T_k(t) \qquad \begin{pmatrix} 0 < t < t_1 \\ 0 < \xi < 1 \end{pmatrix} \quad \left(\alpha_k = \frac{4k+3}{2}\right)$$

where the function $T_k(t)$ is determined by (3.4); furthermore, the functions $\phi_{2k+1}(\xi) = dP_{2k+1}(\xi)/d\xi$, where $P_{2k+1}(\xi)$ is Legendre's polynomial of the first kind, are orthogonal in the interval (0,1) with the weight $(1 - \xi^2)$, and they fulfil the relations [6]

$$\int_0^1 (1-\xi^2)\,\varphi_{2k+1}(\xi)\,\varphi_{2p+1}(\xi)\,d\xi = \begin{cases} 0 & (p \neq k) \\ \frac{(2k+1)(2k+2)}{4k+3} & (p = k = 0, 1, 2, \ldots) \end{cases} \tag{5.4}$$

For the stresses we find

$$\tau_{\xi\varphi}(t,\xi) = -G(1-\xi^2)\frac{\partial\Psi^*}{\partial\xi} = -G(1-\xi^2)\Big\{e^{-3/2t}\sum_{k=1}^{\infty}[A_k \sinh\alpha_k t + \tag{5.5}$$

$$+ C_k \cosh\alpha_k t]\,\varphi'_{2k+1}(\xi) + B_0 P''_{-1/2}(\xi)\,e^{-3/2t} + \sum_{k=1}^{\infty} B_k P''_{-1/2+\mu_k i}(\xi)\,T_k(t)\Big\}$$

$$\tau_{t\varphi}(t,\xi) = G\sqrt{1-\xi^2}\,\frac{\partial\Psi^*}{\partial t} = G\sqrt{1-\xi^2}\Big\{-3A_0 e^{-3t} + \tag{5.6}$$

$$+ \frac{e^{-3/2t}}{2}\sum_{k=1}^{\infty}\Big[[A_k(4k+3) - 3C_k]\cosh\alpha_k t + [C_k(4k+3) - 3A_k]\sinh\alpha_k t\Big]\times$$

$$\times\,\varphi_{2k+1}(\xi) - \frac{3}{2}B_0 P'_{-1/2}(\xi)\,e^{-3/2t} + \sum_{k=1}^{\infty} B_k P'_{-1/2+\mu_k i}(\xi)\,T_k'(t)\Big\}$$

Satisfying condition (5.1), we obtain

$$-G\Big\{e^{-3/2t}\sum_{k=1}^{\infty}[A_k \sinh\alpha_k t + C_k\cosh\alpha_k t]\,\varphi'_{2k+1}(0) + \tag{5.7}$$

$$+ B_0 P''_{-1/2}(0)\,e^{-3/2t} + \sum_{k=1}^{\infty} B_k P''_{-1/2+\mu_k i}(0)\,T_k(t)\Big\} = f_1(t)$$

Since the equations (see [6 ])

$$\varphi'_{2p+1}(\xi) = \frac{d^2}{d\xi^2}P_{2p+1}(\xi) = \frac{1}{2}\sum_{k=1}^{p}(4k-1)(2p-2k+2)(2p+2k+1)P_{2k-1}(\xi)$$

$$P_{2k-1}(0) = 0 \quad (k = 1, 2, \ldots)$$

lead to

$$\varphi'_{2p+1}(0) = 0 \tag{5.9}$$

relation (5.7) assumes the form

$$-G[B_0 P''_{-1/2}(0)\,e^{-3/2t} + \sum_{k=1}^{\infty} B_k P''_{-1/2+\mu_k i}(0)\,T_k(t)] = f_1(t) \tag{5.10}$$

Expanding the function $f_1(t)$ into a series (3.9) in terms of the functions $T_k(t)$, we derive from (5.10)

$$B_0 = -\frac{g_0}{GP''_{-1/2}(0)}, \qquad B_k = -\frac{g_k}{GP''_{-1/2+\mu_k i}(0)} \tag{5.11}$$

Satisfying the first of conditions (5.2), we obtain

$$G\sqrt{1-\xi^2}\left\{-3A_0 + \frac{1}{2}\sum_{k=1}^{\infty}[A_k(4k+3) - 3C_k]\varphi_{2k+1}(\xi) - \right.$$
$$\left. -\frac{3}{2}B_0P'_{-1/2}(\xi) - \frac{3}{2}\sum_{k=1}^{\infty}B_kP'_{-1/2+\mu_k i}(\xi)\right\} = f_2(\xi) \tag{5.12}$$

Expanding the function $f_2(\xi)$ into a series in terms of the functions $(1-\xi^2)^{1/2}\,\phi_{2k+1}(\xi)$, we find [6]

$$f_2(\xi) = a_0\sqrt{1-\xi^2} + \sum_{k=1}^{\infty}a_k(1-\xi^2)^{1/2}\varphi_{2k+1}(\xi) \tag{5.13}$$

where

$$a_0 = \frac{3}{2}\int_0^1\sqrt{1-\xi^2}\,f_2(\xi)\,d\xi \tag{5.14}$$

$$a_k = \frac{4k+3}{(2k+1)(2k+2)}\int_0^1\sqrt{1-\xi^2}\,f_2(\xi)\,\varphi_{2k+1}(\xi)d\xi$$

Multiplying, furthermore, Equation (5.12) by $\sqrt{(1-\xi^2)}$ and integrating with respect to $\xi$ between the limits zero and one, we find

$$-2A_0G + \frac{2}{3}GB_0P_{-1/2}(0) + \frac{3}{2}G\sum_{k=1}^{\infty}B_k\frac{P''_{-1/2+\mu_k i}(0)}{9/4+\mu_k^2} = \frac{2}{3}a_0 \tag{5.15}$$

This result has been obtained with the aid of Equations (5.4) and (5.14)

$$\varphi_1(\xi) = 1$$

$$\int_0^1(1-\xi^2)P'_{-1/2}(\xi)\,d\xi = \frac{1}{9}(\xi^2-1)^2P''_{-1/2}(\xi)\Big|_0^1 = -\frac{4}{9}P''_{-1/2}(0) \tag{5.16}$$

$$\int_0^1(1-\xi^2)P'_{-1/2+\mu_k i}(\xi)\,d\xi = \frac{(\xi^2-1)^2P''_{-1/2+\mu_k i}(\xi)}{9/4+\mu_k^2}\Bigg|_0^1 = -\frac{P''_{-1/2+\mu_k i}(0)}{9/4+\mu_k^2}$$

Substituting the values (5.11) into (5.15) and solving the resulting equation for $A_0$, we find

$$A_0 = -\frac{1}{3G}\left(a_0 + g_0 + \frac{9}{4}\sum_{k=1}^{\infty}\frac{g_k}{{}^9/_4 + \mu_k^2}\right) \tag{5.17}$$

Multiplying Equation (5.12) by $\sqrt{(1-\xi^2)}\phi_{2p+1}(\xi)$ and integrating the resulting relation with respect to $\xi$ between the same limits, we find, with the aid of (5.4) and (5.14)

$$\frac{G\,[A_p\,(4p+3) - 3\,C_p]}{2}\,\frac{(2p+1)\,(2p+2)}{4p+3} - \frac{3}{2}GB_0\int_0^1(1-\xi^2)\,P'_{-1/2}(\xi)\varphi_{2p+1}(\xi)\,d\xi -$$
$$-\frac{3}{2}G\sum_{k=1}^{\infty}B_k\int_0^1(1-\xi^2)\,P'_{-1/2+\mu_k i}(\xi)\,\varphi_{2p+1}(\xi)\,d\xi = \frac{(2p+1)\,(2p+2)}{4p+3}\,a_p \tag{5.18}$$

Let us compute the integrals appearing in (5.18). Since the functions $\phi_{2p+1}(\xi)$ and $P'_{-1/2+\mu_k i}(\xi) = y_k(\xi)$ satisfy the equations

$$(\xi^2-1)\,y_k''(\xi) + 4\xi y_k'(\xi) + (\mu_k^2 + {}^9/_4)\,y_k(\xi) = 0 \tag{5.19}$$
$$(\xi^2-1)\,\varphi''_{2p+1}(\xi) + 4\xi\varphi'_{2p+1}(\xi) - (4p^2+6p)\,\varphi_{2p+1}(\xi) = 0$$

we proceed as follows: multiplying the first of these equations by $(\xi^2-1)\phi_{2p+1}(\xi)$ and the second by $(\xi^2-1)y_k(\xi)$, and subtracting the second from the first, we obtain the relation

$$\frac{d}{d\xi}\{(\xi^2-1)^2\,[\varphi_{2p+1}(\xi)\,y_k'(\xi) - \varphi'_{2p+1}(\xi)\,y_k(\xi)]\} =$$
$$= \left[\mu_k^2 + \left(\frac{3}{2}+2p\right)^2\right](1-\xi^2)\,\varphi_{2p+1}(\xi)\,y_k(\xi) \tag{5.20}$$

Integrating the latter with respect to $\xi$ between the limits zero and one, we have, with the aid of (5.9)

$$\int_0^1(1-\xi^2)\,P'_{-1/2+\mu_k i}(\xi)\,\varphi_{2p+1}(\xi)\,d\xi$$
$$= \frac{(\xi^2-1)^2}{\mu_k^2 + ({}^3/_2+2p)^2}\,[\varphi_{2p+1}(\xi)\,P''_{-1/2+\mu_k i}(\xi) - \varphi'_{2p+1}(\xi)\,P'_{-1/2+\mu_k i}(\xi)]\Big|_0^1 \tag{5.21}$$
$$= -\frac{P''_{-1/2+\mu_k i}(0)}{\mu_k^2 + ({}^3/_2+2p)^2}\,\varphi_{2p+1}(0)$$

We note, furthermore, that [6 ]

$$\varphi_{2p+1}(\xi) = \frac{d}{d\xi} P_{2p+1}(\xi) = \sum_{k=0}^{p} (4k+1) P_{2k}(\xi) \tag{5.22}$$

$$P_{2k}(0) = (-1)^k \frac{1\cdot 3\cdot 5 \ldots (2k-1)}{2\cdot 4\cdot 6 \ldots 2k}. \tag{5.23}$$

Therefore

$$\varphi_{2p+1}(0) = \sum_{k=0}^{p} (-1)^k \frac{1\cdot 3\cdot 5 \ldots (2k-1)}{2\cdot 4\cdot 6 \ldots 2k} (4k+1) < \frac{p+2}{2} \tag{5.24}$$

For the other integral we find the value

$$\int_0^1 (1-\xi^2) P'_{-1/2}(\xi) \varphi_{2p+1}(\xi) d\xi = -\frac{4P''_{-1/2}(0)}{(4p+3)^2} \varphi_{2p+1}(0) \tag{5.25}$$

Substituting the values (5.21) and (5.25) into (5.18), we find

$$A_p(4p+3) - 3C_p = \frac{2}{G} Q_p \tag{5.26}$$

where (5.27)

$$Q_p = a_p + \frac{6\varphi_{2p+1}(0)}{(2p+1)(2p+2)(4p+3)} g_0 + \frac{3(4p+3)\varphi_{2p+1}(0)}{2(2p+1)(2p+2)} \sum_{k=1}^{\infty} \frac{g_k}{\mu_k^2 + (3/2+2p)^2}$$

Analogously, satisfying the second of conditions (5.2), we obtain the relations

$$A_0 = -\frac{1}{3G}\left[ b_0 e^{3t_1} + g_0 e^{3/2 t_1} + \frac{9}{4} e^{3/2 t_1} \sum_{k=1}^{\infty} \frac{(-1)^k g_k}{9/4 + \mu_k^2}\right] \tag{5.28}$$

$$A_p[(4p+3)\cosh\alpha_p t_1 - 3\sinh\alpha_p t_1] + C_p[(4p+3)\sinh\alpha_p t_1 - 3\cosh\alpha_p t_1] = \frac{2}{G} N_p \tag{5.29}$$

where the notations

$$b_0 = \frac{3}{2}\int_0^1 \sqrt{1-\xi^2} f_3(\xi) d\xi$$

$$b_k = \frac{4k+3}{(2k+1)(2k+2)} \int_0^1 \sqrt{1-\xi^2} f_3(\xi) \varphi_{2k+1}(\xi) d\xi \tag{5.30}$$

$$N_p = b_p e^{3/2 t_1} + \frac{6\varphi_{2p+1}(0)}{(2p+1)(2p+2)(4p+3)} g_0 + \tag{5.31}$$

$$+ \frac{3(4p+3)\varphi_{2p+1}(0)}{2(2p+1)(2p+2)} \sum_{k=1}^{\infty} \frac{(-1)^k g_k}{\mu_k^2 + (3/2+2p)^2}$$

are used.

Combining Expressions (5.17) and (5.28), we obtain the following relation:

$$a_0 - b_0 e^{3t_1} + g_0 (1 - e^{3/2 t_1}) + \frac{9}{4} \sum_{k=1}^{\infty} \frac{g_k [1 + (-1)^{k+1} e^{3/2 t_1}]}{\mu_k^2 + 9/4} = 0 \qquad (5.32)$$

This is the equation of equilibrium for the torques twisting the shaft.

Solving Equations (5.26) and (5.29) for $A_p$ and $C_p$, we find

$$A_p = \frac{1}{4G(2p+3)p} \{Q_p [4p + 3 - 3 \coth \alpha_p t_1] + 3N_p \operatorname{cosech} \alpha_p t_1\} \qquad (5.33)$$

$$C_p = \frac{1}{4G(2p+3)p} \{(4p+3) N_p \operatorname{cosech} \alpha_p t_1 - Q_p [(4p+3) \coth \alpha_p t_1 - 3]\}$$

Finally, substituting the values (5.11), (5.17) and (5.33) into (5.3), we obtain the following expression for the displacement function $\Psi^*(t, \xi)$:

$$\Psi^*(t, \xi) = -\frac{e^{-3t}}{3G}\left(a_0 + g_0 + 9/4 \sum_{k=1}^{\infty} \frac{g_k}{9/4 + \mu_k^2}\right) +$$

$$+ \frac{e^{-3/2 t}}{4G} \sum_{p=1}^{\infty} \frac{\varphi_{2p+1}(\xi)}{(2p+3)p} \{Q_p [3 \sinh \alpha_p (t_1 - t) \operatorname{cosech} \alpha_p t_1 -$$

$$- (4p+3) \cosh \alpha_p (t_1 - t) \operatorname{cosech} \alpha_p t_1] - N_p [3 \sinh \alpha_p t \operatorname{cosech} \alpha_p t_1 +$$

$$+ (4p+3) \cosh \alpha_p t \operatorname{cosech} \alpha_p t_1]\} - \frac{g_0 e^{-3/2 t}}{G} \frac{P'_{-1/2}(\xi)}{P''_{-1/2}(0)} -$$

$$- \frac{e^{-3/2 t}}{G} \sum_{k=1}^{\infty} g_k \frac{P'_{-1/2 + \mu_k i}(\xi)}{P''_{-1/2 + \mu_k i}(0)} \cos \frac{k \pi t}{t_1} \qquad (5.34)$$

It is easily seen that the series occurring in Expression (5.34) are convergent.

Using Formulas (1.8) and (5.34), we can determine the stresses at any arbitrary point of the axial section of the hemisphere.

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